Suppose we make people more productive at a subset of tasks, and we see how they reallocate their time, what can we infer about their *overall* productivity?

We're going to assume (1) they are optimizing some unobserved objective functionm $y(\cdot)$; (2) that function has constant returns to scale (if they doubled their time on every task, they'd double output).

The following results are all adaptions of very old standard economic theory about inferring the effects of a *price* change on economic welfare.

Formally: suppose someone is allocating their time across N different tasks, $t_1, \ldots, t_N$, in order to maximize some function $y(A_1 t_1, \ldots, A_N t_N)$. Each

$$
\begin{aligned}
y(A_1 t_1, \ldots, A_n t_n) \quad & \text{(output)} \\
V(A) \equiv \max_{t \in \mathbb{R}^n_+} y(A_1 t_1, \ldots, A_n t_n) \quad & \text{(value)} \\
\sum_{i=1}^{N} t_i = 1 \quad & \text{(time allocation before AI)} \\
\sum_{i=1}^{N} t_i' = 1 \quad & \text{(time allocation after AI)} \\
m_i \equiv \frac{A_i'}{A_i} \quad & \text{(productivity changes due to AI)}
\end{aligned}
$$

What you observe / assume	What we know about uplift ($\frac{V'}{V}$)	Reference
You know $y(\cdot)$, $\boldsymbol{m}$ and $\boldsymbol{t}$	Exact V'/V	Prop. 3
You know m and t, and the person cannot adjust their time allocations (so $\boldsymbol{t}' = \boldsymbol{t}$)	$m_{\min} \leq \frac{y(A' \circ t)}{y(A \circ t)} \leq m_{\max}$	Prop. 3.1
You know $\boldsymbol{m}$ but not $\boldsymbol{t}$ or $\boldsymbol{t}'$	$m_{\min} \leq V'/V \leq m_{\max}$	Cor. 4.1 (part 3)
You know t and multipliers m (but not t')	$\left(\sum_i \frac{t_i}{m_i}\right)^{-1} \leq \frac{V'}{V} \leq m_{\max}$	Cor. 4.1 (part 1)
You know t' and multipliers m (but not t)	$m_{\min} \leq \frac{V'}{V} \leq \sum_i t_i' m_i$	Cor. 4.1 (part 2)
You know t and t' and m	$\left(\sum_i \frac{t_i}{m_i}\right)^{-1} \leq \frac{V'}{V} \leq \sum_i t_i' m_i$	Prop. 4
You know t and m, and $\boldsymbol{m} \simeq 1$	$\ln \frac{V'}{V} \approx \sum_i t_i \ln m_i$	Cor. 7.1
Large changes, know a path $A(\tau)$ and shares along it	$\ln \frac{V'}{V} = \int_0^1 \sum_i t_i(A(\tau)) \frac{d}{d\tau} \ln A_i(\tau)\, d\tau$	Prop. 8
You assume $y(\cdot)$ is CES, $n = 2$, only task 2 multiplied by $A_2^{(m)}$, know baseline t_2	$\frac{V'}{V} = \left((1 - t_2) + t_2 (A_2^{(m)})^{\varepsilon - 1}\right)^{\frac{1}{\varepsilon - 1}}$	Prop. 11
You assume $y(\cdot)$ is CES, $n = 2$, observe $t_2, t_2', A_2^{(m)}$	$\varepsilon = 1 + \frac{\text{logit}(t_2') - \text{logit}(t_2)}{\ln A_2^{(m)}}$ (& use Prop 11 for $\frac{V'}{V}$)	Prop. 12 (+ Proposition 11)

Formal derivations

Assumptions

Assumption A1 (Tasks and time endowment). There are $n \in \mathbb{N}$ tasks. A time allocation is a vector $t = (t_1, \dots, t_n) \in \mathbb{R}^n_+$ satisfying

$$\sum_{i=1}^{n} t_i \leq 1.$$

Assumption A2 (Task productivities). A productivity vector is $A = (A_1, \dots, A_n) \in \mathbb{R}^n_{++}$. Effective task outputs are

$$z_i \equiv A_i t_i, \qquad i = 1, \dots, n.$$

Assumption A3 (Output aggregator). The function $y : \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous and weakly increasing in each coordinate. Moreover, (y) is concave and (for duality and index-number identities) **linearly homogeneous**:

$$y(\lambda z) = \lambda y(z) \quad \text{for all } \lambda \geq 0,\ z \in \mathbb{R}^n_+.$$

Assumption A4 (Differentiability for share formulas). When required, (y) is differentiable on $\mathbb{R}^n_{++}$, and the associated unit-expenditure index (defined below) is differentiable on $\mathbb{R}^n_{++}$.

Definitions

Definition D1 (Primal productivity level). Define the maximal output attainable under productivity (A) by

$$V(A) \equiv \max_{t \in \mathbb{R}^n_+} y(A_1 t_1, \dots, A_n t_n) \quad \text{s.t.} \quad \sum_{i=1}^{n} t_i \leq 1.$$

Definition D2 (Time prices). Define the vector of time prices $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{++}$ by

$$p_i \equiv \frac{1}{A_i}.$$

Definition D3 (Time-price (consumption-form) problem). Define the equivalent problem

$$\tilde{V}(p) \equiv \max_{z \in \mathbb{R}^n_+} y(z) \quad \text{s.t.} \quad \sum_{i=1}^{n} p_i z_i \leq 1.$$

Definition D4 (Expenditure function and unit-expenditure index). For $u \geq 0$, define

$$e(p, u) \equiv \min_{z \in \mathbb{R}^n_+} p \cdot z \quad \text{s.t.} \quad y(z) \geq u.$$

Define the unit-expenditure index

$$P(p) \equiv e(p, 1).$$

Definition D5 (Hicksian (compensated) demand and shares). Let $(h(p, u))$ denote any minimizer in the definition of $(e(p, u))$. Define Hicksian shares at $((p, u))$ by

$$s_i^H(p, u) \equiv \frac{p_i h_i(p, u)}{e(p, u)}.$$

When $u = 1$, write $s_i^H(p) \equiv s_i^H(p, 1)$.

Equivalence and duality

Proposition 1 (Equivalence of the time-allocation and time-price formulations) For every $A \in \mathbb{R}^n_{++}$ and $p = 1/A$,

$$V(A) = \tilde{V}(p).$$

Proof

1. Fix $A \in \mathbb{R}^n_{++}$ and define $p_i = 1/A_i$ for all (i).
2. For any feasible (t) in Definition D1, define $z_i \equiv A_i t_i$. Then $z \in \mathbb{R}^n_+$ and

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} \frac{1}{A_i}, (A_i t_i) = \sum_{i=1}^{n} t_i \leq 1,$$

   so (z) is feasible for Definition D3. Moreover $y(A \circ t) = y(z)$. Hence $V(A) \leq \tilde{V}(p)$.
3. Conversely, for any feasible (z) in Definition D3, define $t_i \equiv p_i z_i = z_i/A_i$. Then $t \in \mathbb{R}^n_+$ and $\sum_i t_i = \sum_i p_i z_i \leq 1$, so (t) is feasible for Definition D1. Moreover $A_i t_i = z_i$, hence $y(A \circ t) = y(z)$. Thus $\tilde{V}(p) \leq V(A)$.
4. By 2 and 3, $V(A) = \tilde{V}(p)$. □

Proposition 2 (Productivity level as the reciprocal of a unit-expenditure index). Under Assumption A3, for every $p \in \mathbb{R}^n_{++}$,

$$\tilde{V}(p) = \frac{1}{P(p)}.$$

Equivalently, for every $A \in \mathbb{R}^n_{++}$,

$$V(A) = \frac{1}{P(1/A)}.$$

Proof

1. Fix $p \in \mathbb{R}^n_{++}$. Consider the set $z \geq 0 \,:\, p \cdot z \leq 1$. By linear homogeneity of (y), for any $\lambda > 0$,

$$y(\lambda z) = \lambda y(z).$$

2. Let $z^\star$ attain $\tilde{V}(p)$ in Definition D3, and set $u^\star \equiv y(z^\star) = \tilde{V}(p)$. Then $p \cdot z^\star \leq 1$.
3. Define $\hat{z} \equiv z^\star/u^\star$. By linear homogeneity, $y(\hat{z}) = 1$. Hence $\hat{z}$ is feasible for $P(p) = e(p, 1)$.
4. Compute its cost:

$$p \cdot \hat{z} = p \cdot (z^\star/u^\star) = (p \cdot z^\star)/u^\star \leq 1/u^\star.$$

   Since (P(p)) is the minimum cost to reach $y(\cdot) \geq 1$, 4 implies

$$P(p) \leq \frac{1}{u^\star} = \frac{1}{\tilde{V}(p)}.$$

   Equivalently, $\tilde{V}(p) \leq 1/P(p)$.
5. Conversely, let z^H attain (P(p)) with $y(z^H) \geq 1$. By monotonicity of (y), one may take $y(z^H) = 1$. Define $\bar{z} \equiv z^H/P(p)$. Then $p \cdot \bar{z} = 1$ and, by linear homogeneity, $y(\bar{z}) = 1/P(p)$. Hence $\bar{z}$ is feasible in Definition D3 and achieves $y(\bar{z}) = 1/P(p)$, so $\tilde{V}(p) \geq 1/P(p)$.
6. Combine 4–5 to conclude $\tilde{V}(p) = 1/P(p)$. Substituting $p = 1/A$ yields $V(A) = 1/P(1/A)$. □

Exact productivity indices and revealed-preference bounds

Proposition 3 (Exact productivity index between two productivity vectors). Let $A, A' \in \mathbb{R}^n_{++}$ and set $p = 1/A$, $p' = 1/A'$. Under Assumption A3,

$$\frac{V(A')}{V(A)} = \frac{P(p)}{P(p')}.$$

Proof

1. By Proposition 2, $V(A) = 1/P(p)$ and $V(A') = 1/P(p')$.
2. Therefore,
$$\frac{V(A')}{V(A)} = \frac{1/P(p')}{1/P(p)} = \frac{P(p)}{P(p')}.$$
   □

Proposition 3.1 (Effect of productivity multipliers holding time allocation fixed). Let $A, A' \in \mathbb{R}^n_{++}$ and define multipliers $m_i \equiv A'_i/A_i$. Fix any feasible time allocation $t \in \mathbb{R}^n_+$ with $\sum_i t_i \leq 1$ and define realized output

$$Y(A; t) \equiv y(A_1 t_1, \dots, A_n t_n).$$

Let $m_{\min} \equiv \min_i m_i$ and $m_{\max} \equiv \max_i m_i$. Under Assumption A3,

$$m_{\min} \leq \frac{Y(A'; t)}{Y(A; t)} \leq m_{\max}.$$

Proof

1. Let $z \equiv A \circ t \in \mathbb{R}^n_+$ denote the effective-output vector under (A, t), so $Y(A; t) = y(z)$.
2. Since $A' = m \circ A$, we have $A' \circ t = (m \circ A) \circ t = m \circ (A \circ t) = m \circ z$. Hence $Y(A'; t) = y(m \circ z)$.
3. By definition of $m_{\min}, m_{\max}$, we have the coordinatewise bounds $m_{\min} z \leq m \circ z \leq m_{\max} z$.
4. By weak monotonicity of y (A3), $y(m_{\min} z) \leq y(m \circ z) \leq y(m_{\max} z)$.
5. By linear homogeneity of y (A3), $y(m_{\min} z) = m_{\min} y(z)$ and $y(m_{\max} z) = m_{\max} y(z)$.
6. Substitute 1–5 and divide by $y(z) = Y(A; t)$ to obtain $m_{\min} \leq Y(A'; t)/Y(A; t) \leq m_{\max}$. □

Proposition 4 (Laspeyres–Paasche bounds from observing baseline and/or post allocations). Let $A, A' \in \mathbb{R}^n_{++}$ with $p = 1/A$, $p' = 1/A'$. Assume A3. Let $(t(A))$ and $(t(A'))$ denote optimal time shares in Definition D1. Define multipliers

$$m_i \equiv \frac{A'_i}{A_i} \qquad (i = 1, \dots, n).$$

Define baseline and post shares by $t_i \equiv t_i(A)$ and $t'_i \equiv t_i(A')$.

Then the exact productivity index satisfies

$$\boxed{\frac{1}{\sum_{i=1}^n t_i \frac{1}{m_i}} \leq \frac{V(A')}{V(A)} \leq \sum_{i=1}^n t'_i m_i}$$

where the lower bound uses only the baseline shares (t), and the upper bound uses only the post shares (t').

Proof

1. Let $u = 1$. Let (h(p,1)) and (h(p',1)) be Hicksian demands for unit output under prices (p) and (p'). Then
$$P(p) = p \cdot h(p,1), \qquad P(p') = p' \cdot h(p',1).$$
2. Since (h(p,1)) is feasible (achieves output (1)), it provides an upper bound on the minimum cost at prices (p'):
$$P(p') = \min p' \cdot z \ : \ \ y(z) \geq 1 \leq p' \cdot h(p,1).$$
Divide by $P(p) = p \cdot h(p,1)$ to obtain
$$\frac{P(p')}{P(p)} \leq \frac{p' \cdot h(p,1)}{p \cdot h(p,1)}.$$
3. Similarly, since (h(p',1)) is feasible, it provides an upper bound on the minimum cost at prices (p):
$$P(p) = \min p \cdot z \ : \ \ y(z) \geq 1 \leq p \cdot h(p',1).$$
Rearrange to obtain
$$\frac{P(p')}{P(p)} \geq \frac{p' \cdot h(p',1)}{p \cdot h(p',1)}.$$
4. Combine 2–3 and invert to bound the productivity index $\Gamma \equiv V(A')/V(A) = P(p)/P(p')$ (Proposition 3):
$$\frac{1}{\frac{p' \cdot h(p,1)}{p \cdot h(p,1)}} \leq \Gamma \leq \frac{1}{\frac{p' \cdot h(p',1)}{p \cdot h(p',1)}}.$$
5. Express the right-hand sides in shares. For unit output, the Hicksian share is
$$s_i^H(p) = \frac{p_i h_i(p,1)}{P(p)}, \qquad s_i^H(p') = \frac{p_i' h_i(p',1)}{P(p')}.$$
Thus
$$\frac{p' \cdot h(p,1)}{p \cdot h(p,1)} = \sum_{i=1}^{n} s_i^H(p), \frac{p_i'}{p_i}, \qquad \frac{p' \cdot h(p',1)}{p \cdot h(p',1)} = \frac{1}{\sum_{i=1}^{n} s_i^H(p'), \frac{p_i}{p_i'}}.$$
6. By Proposition 6 below (time shares coincide with Hicksian shares at the optimum under the unit time endowment), $s_i^H(p) = t_i$ and $s_i^H(p') = t_i'$. Moreover $\frac{p_i'}{p_i} = \frac{A_i}{A_i'} = \frac{1}{m_i}$ and $\frac{p_i}{p_i'} = m_i$. Substitute into 5 and then into 4 to obtain the stated bounds. □

Corollary 4.1 (Bounds when only one allocation is observed). Under the assumptions and notation of Proposition 4, define
$$m_{\min} \equiv \min_i m_i, \qquad m_{\max} \equiv \max_i m_i.$$
Then the gain $\Gamma \equiv V(A')/V(A)$ satisfies:

1. **If you observe** (t, m) **but not** t', then
$$\left(\sum_i \frac{t_i}{m_i}\right)^{-1} \leq \Gamma \leq m_{\max}.$$
2. **If you observe** (t', m) **but not** t, then
$$m_{\min} \leq \Gamma \leq \sum_i t_i' m_i.$$
3. **If you observe only** m **(no allocations)**, then
$$m_{\min} \leq \Gamma \leq m_{\max}.$$

Proof

1. Let $m \equiv (m_1, \dots, m_n)$ and note that for any $z \in \mathbb{R}^n_+$,

$$m_{\min} z \le m \circ z \le m_{\max} z$$

   coordinatewise.
2. Since $y(\cdot)$ is weakly increasing (A3), this implies

$$y(m_{\min} z) \le y(m \circ z) \le y(m_{\max} z).$$

3. By linear homogeneity (A3), $y(m_{\min} z) = m_{\min} y(z)$ and $y(m_{\max} z) = m_{\max} y(z)$, hence

$$m_{\min} y(z) \le y(m \circ z) \le m_{\max} y(z).$$

4. Apply 3 with $z = A \circ t$ for any feasible time allocation t to obtain

$$m_{\min} y(A \circ t) \le y(A' \circ t) \le m_{\max} y(A \circ t),$$

   since $A' \circ t = (m \circ A) \circ t = m \circ (A \circ t)$.
5. Maximize over feasible t and use Definition D1 to conclude

$$m_{\min} V(A) \le V(A') \le m_{\max} V(A),$$

   which gives $m_{\min} \le \Gamma \le m_{\max}$ (part 3).
6. Part 1 combines the lower bound from Proposition 4 (which uses only t and m) with $\Gamma \le m_{\max}$ from part 3.
7. Part 2 combines the upper bound from Proposition 4 (which uses only t' and m) with $\Gamma \ge m_{\min}$ from part 3.
   □

Welfare in time units (EV and CV)

Proposition 5 (Equivalent and compensating variation measured in time). Let $A, A' \in \mathbb{R}^n_{++}$ with $p = 1/A$, $p' = 1/A'$. Define

$$EV \equiv e\big(p, V(A')\big) - 1, \qquad CV \equiv e\big(p', V(A)\big) - 1.$$

Under Assumption A3,

$$EV = \frac{P(p)}{P(p')} - 1, \qquad CV = \frac{P(p')}{P(p)} - 1.$$

Proof

1. Under Assumption A3 (linear homogeneity), the expenditure function is homogeneous of degree (1) in the required output level:

$$e(p, u) = u, e(p, 1) = u, P(p).$$

2. By Proposition 2, $V(A') = 1/P(p')$ and $V(A) = 1/P(p)$.
3. Compute

$$EV = e(p, V(A')) - 1 = V(A'), P(p) - 1 = \frac{1}{P(p')}, P(p) - 1 = \frac{P(p)}{P(p')} - 1.$$

4. Similarly,

$$CV = e(p', V(A)) - 1 = V(A), P(p') - 1 = \frac{1}{P(p)}, P(p') - 1 = \frac{P(p')}{P(p)} - 1.$$

   □

Share formulas: differentials, exact integrals, and approximations

Proposition 6 (Time shares coincide with Hicksian shares). Assume A3 and let $(t(A))$ be optimal in Definition D1. Let $p = 1/A$. Then the Hicksian shares at unit output satisfy

$$s_i^H(p) = t_i(A) \qquad (i = 1, \dots, n).$$

Proof

1. Let $z^\star$ solve Definition D3 under prices (p), and let $\tilde{V}(p) = y(z^\star)$. By Proposition 1, there exists an optimal (t(A)) with $z_i^\star = A_i t_i(A)$.
2. Under monotonicity of y, the budget constraint binds at the optimum in Definition D3: $p \cdot z^\star = 1$. (Otherwise a uniform expansion of $z^\star$ would raise y without violating the constraint.)
3. Consider the unit-output Hicksian bundle (h(p,1)). By Proposition 2, $\tilde{V}(p) = 1/P(p)$. By linear homogeneity, the scaled bundle $\hat{z} \equiv z^\star/\tilde{V}(p) = z^\star, P(p)$ satisfies $y(\hat{z}) = 1$. It is feasible for $P(p) = e(p, 1)$ and has cost

$$p \cdot \hat{z} = p \cdot (z^\star P(p)) = (p \cdot z^\star) P(p) = 1 \cdot P(p) = P(p).$$

   Hence $\hat{z}$ attains the minimum in (e(p,1)), so $h(p, 1) = \hat{z} = z^\star P(p)$.
4. Compute Hicksian shares:

$$s_i^H(p) = \frac{p_i h_i(p, 1)}{P(p)} = \frac{p_i (z_i^\star P(p))}{P(p)} = p_i z_i^\star.$$

5. Substitute $p_i = 1/A_i$ and $z_i^\star = A_i t_i(A)$ to obtain $p_i z_i^\star = t_i(A)$. Therefore $s_i^H(p) = t_i(A)$. □

Proposition 7 (Differential share representation). Assume A3–A4. Then

$$d \ln P(p) = \sum_{i=1}^{n} s_i^H(p), d \ln p_i.$$

Equivalently, with $p = 1/A$,

$$d \ln V(A) = \sum_{i=1}^{n} t_i(A) d \ln A_i.$$

Proof

1. By Shephard's lemma under Assumption A4, for $u = 1$,

$$\frac{\partial P(p)}{\partial p_i} = h_i(p, 1).$$

2. Compute the differential:

$$dP(p) = \sum_{i=1}^{n} \frac{\partial P(p)}{\partial p_i}, d p_i = \sum_{i=1}^{n} h_i(p, 1), d p_i.$$

3. Divide by (P(p)) and rewrite in log differentials:

$$d \ln P(p) = \frac{dP(p)}{P(p)} = \sum_{i=1}^{n} \frac{h_i(p, 1), p_i}{P(p)}, d \ln p_i = \sum_{i=1}^{n} s_i^H(p), d \ln p_i.$$

4. By Proposition 2, $\ln V(A) = -\ln P(1/A)$. Hence

$$d\ln V(A) = -d\ln P(p) \quad \text{with } p_i = 1/A_i,$$

and $d\ln p_i = -d\ln A_i$. Therefore

$$d\ln V(A) = \sum_{i=1}^{n} s_i^H(p), d\ln A_i.$$

5. Apply Proposition 6 to substitute $s_i^H(p) = t_i(A)$. □

Corollary 7.1 (First-order approximation using baseline time shares).

For a small change $A \to A'$,

$$\ln \frac{V(A')}{V(A)} = \sum_{i=1}^{n} t_i(A) \ln\left(\frac{A_i'}{A_i}\right) + o\left(\|A' - A\|\right).$$

Proof

Immediate from Proposition 7 by evaluating $t_i(\cdot)$ at (A) and applying a first-order expansion. □

Proposition 8 (Exact integral representation for large changes). Assume A3–A4. Let $p(\tau)$ be a differentiable path in $\mathbb{R}_{++}^n$ with $p(0) = p$ and $p(1) = p'$. Then

$$\ln \frac{P(p')}{P(p)} = \int_0^1 \sum_{i=1}^{n} s_i^H\big(p(\tau)\big), \frac{d}{d\tau} \ln p_i(\tau) d\tau.$$

Consequently, for $A(\tau) = 1/p(\tau)$,

$$\ln \frac{V(A')}{V(A)} = \int_0^1 \sum_{i=1}^{n} t_i\big(A(\tau)\big), \frac{d}{d\tau} \ln A_i(\tau) d\tau.$$

Proof

1. By Proposition 7, for each τ,

$$\frac{d}{d\tau} \ln P\big(p(\tau)\big) = \sum_{i=1}^{n} s_i^H\big(p(\tau)\big), \frac{d}{d\tau} \ln p_i(\tau).$$

2. Integrate both sides from $\tau = 0$ to $\tau = 1$:

$$\ln P(p') - \ln P(p) = \int_0^1 \sum_{i=1}^{n} s_i^H\big(p(\tau)\big), \frac{d}{d\tau} \ln p_i(\tau) d\tau.$$

3. Substitute $V(A) = 1/P(1/A)$ and $s_i^H(1/A) = t_i(A)$ (Propositions 2 and 6), and use $\ln p_i(\tau) = -\ln A_i(\tau)$. □

Corollary 8.1 (Single-component change). Assume only p_j varies along the path and all p_{-j} are constant. Then

$$\ln \frac{P(p')}{P(p)} = \int_{\ln p_j}^{\ln p_j'} s_j^H\big(p_j\big), d(\ln p_j).$$

Equivalently,

$$\ln \frac{V(A')}{V(A)} = \int_{\ln A_j}^{\ln A_j'} t_j\big(A_j\big), d(\ln A_j),$$

holding A_{-j} fixed.

Proof

Specialize Proposition 8 to paths with only one varying coordinate. □

Proposition 9 Törnqvist/Divisia trapezoid approximation. Assume A3–A4. For a finite change $p \to p'$, define average shares $\bar{s}_i \equiv \frac{1}{2}(s_i^H(p) + s_i^H(p'))$. Then the trapezoid approximation to Proposition 8 yields

$$\ln \frac{P(p')}{P(p)} \approx \sum_{i=1}^{n} \bar{s}_i, \ln\left(\frac{p_i'}{p_i}\right).$$

Equivalently, defining $\bar{t}_i \equiv \frac{1}{2}(t_i(A) + t_i(A'))$,

$$\ln \frac{V(A')}{V(A)} \approx \sum_{i=1}^{n} \bar{t}_i, \ln\left(\frac{A_i'}{A_i}\right).$$

Proof

1. By Proposition 8,
$$\ln \frac{P(p')}{P(p)} = \int_0^1 \sum_{i=1}^{n} s_i^H\big(p(\tau)\big), \frac{d}{d\tau} \ln p_i(\tau) d\tau$$
for any differentiable path $p(\tau)$ from (p) to (p').
2. Choose the log-linear path $\ln p_i(\tau) = (1-\tau)\ln p_i + \tau \ln p_i'$. Then $\frac{d}{d\tau} \ln p_i(\tau) = \ln(p_i'/p_i)$ is constant in τ.
3. Under this path,
$$\ln \frac{P(p')}{P(p)} = \sum_{i=1}^{n} \left(\int_0^1 s_i^H\big(p(\tau)\big), d\tau\right) \ln\left(\frac{p_i'}{p_i}\right).$$
4. Approximate $\int_0^1 s_i^H(p(\tau)), d\tau$ by the trapezoid rule:
$$\int_0^1 s_i^H\big(p(\tau)\big), d\tau \approx \frac{1}{2}\big(s_i^H(p) + s_i^H(p')\big) = \bar{s}_i.$$
Substitute into 3.
5. The (V)-form follows from $p_i'/p_i = (A_i/A_i')$ and $s_i^H(1/A) = t_i(A)$. □

CES specialization and the two-good reduction

Assumption C1 (CES aggregator)

For parameters $\sigma > 0$ and weights $\alpha_i > 0$, define

$$y(z) = \left(\sum_{i=1}^{n} \alpha_i, z_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \qquad (\sigma \neq 1),$$

with the $\sigma = 1$ case defined by continuity (Cobb–Douglas).

Proposition 10 (CES unit-expenditure index and CES Hicksian shares). Under Assumption C1, the unit-expenditure index is

$$P(p) = \left(\sum_{i=1}^{n} \alpha_i^{\sigma}, p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}},$$

and Hicksian shares at unit output are

$$s_i^H(p) = \frac{\alpha_i^\sigma, p_i^{,1-\sigma}}{\sum_{j=1}^n \alpha_j^\sigma, p_j^{,1-\sigma}}.$$

Proof

1. Consider the cost-minimization problem defining $P(p) = e(p, 1)$:

$$\min_{z\geq 0} p \cdot z \quad \text{s.t.} \quad \left(\sum_{i=1}^n \alpha_i, z_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \geq 1.$$

   At optimum, the constraint binds.
2. Form the Lagrangian (with multiplier $\lambda > 0$):

$$\mathscr{L}(z, \lambda) = \sum_{i=1}^n p_i z_i + \lambda\left(1 - \left(\sum_{i=1}^n \alpha_i, z_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}\right).$$

3. First-order conditions (interior case) imply that for all (i),

$$p_i \propto \alpha_i, z_i^{-\frac{1}{\sigma}},$$

   hence $z_i \propto (\alpha_i/p_i)^\sigma$.
4. Substitute $z_i = k, (\alpha_i/p_i)^\sigma$ into the binding constraint to solve for (k), and then compute $p \cdot z$. The resulting minimum cost equals

$$P(p) = \left(\sum_{i=1}^n \alpha_i^\sigma, p_i^{,1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

5. Apply Shephard's lemma $h_i(p, 1) = \partial P(p)/\partial p_i$ and normalize shares:

$$s_i^H(p) = \frac{p_i h_i(p, 1)}{P(p)} = \frac{\alpha_i^\sigma, p_i^{,1-\sigma}}{\sum_{j=1}^n \alpha_j^\sigma, p_j^{,1-\sigma}}.$$

□

Proposition 11 (Two-good CES gain from a single productivity multiplier).

Let $n = 2$. Fix $A_1' = A_1$ and let $A_2' = A_2 A_2^{(m)}$ where the multiplier $A_2^{(m)} > 0$ applies only to task 2 (equivalently $p_2' = p_2/A_2^{(m)}$ and $p_1' = p_1$). Let the baseline time share on task 2 be $t_2 \equiv t_2(A)$. Under Assumption C1 with elasticity $\varepsilon \equiv \sigma$,

$$\frac{V(A')}{V(A)} = \left((1 - t_2) + t_2, (A_2^{(m)})^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}}.$$

Proof

1. By Proposition 3,

$$\frac{V(A')}{V(A)} = \frac{P(p)}{P(p')}.$$

2. By Proposition 10 (two-good case),

$$P(p) = \left(\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon p_2^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}, \quad P(p') = \left(\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon (p_2/A_2^{(m)})^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}.$$

3. Define the baseline CES share
$$t_2 \equiv s_2^H(p) = \frac{\alpha_2^\varepsilon p_2^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon p_2^{1-\varepsilon}}.$$
By Proposition 6, this equals the baseline optimal time share.
4. Note $(p_2/A_2^{(m)})^{1-\varepsilon} = p_2^{1-\varepsilon}(A_2^{(m)})^{\varepsilon-1}$. Hence
$$\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon (p_2/A_2^{(m)})^{1-\varepsilon} = \left(\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon p_2^{1-\varepsilon}\right)\left((1-t_2) + t_2 (A_2^{(m)})^{\varepsilon-1}\right).$$
5. Substitute 4 into $P(p)/P(p')$ and simplify to obtain
$$\frac{V(A')}{V(A)} = \left((1-t_2) + t_2, (A_2^{(m)})^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}}.$$

□

Proposition 12 (CES share response and identification from pre/post shares).

In the setting of Proposition 11, let $t_2' \equiv t_2(A')$ denote the post-change time share on task 2. Then
$$t_2' = \frac{t_2, (A_2^{(m)})^{\varepsilon-1}}{(1-t_2) + t_2, (A_2^{(m)})^{\varepsilon-1}},$$
and equivalently,
$$\mathrm{logit}(t_2') - \mathrm{logit}(t_2) = (\varepsilon - 1)\ln A_2^{(m)}, \qquad \mathrm{logit}(x) \equiv \ln\left(\frac{x}{1-x}\right).$$
Thus, if t_2, t_2', and $A_2^{(m)}$ are observed,
$$\varepsilon = 1 + \frac{\mathrm{logit}(t_2') - \mathrm{logit}(t_2)}{\ln A_2^{(m)}}.$$

Proof

1. By Proposition 10 (CES shares),
$$t_2 = s_2^H(p) = \frac{\alpha_2^\varepsilon p_2^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon p_2^{1-\varepsilon}}, \quad t_2' = s_2^H(p') = \frac{\alpha_2^\varepsilon (p_2/A_2^{(m)})^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon} + \alpha_2^\varepsilon (p_2/A_2^{(m)})^{1-\varepsilon}}.$$
2. Substitute $(p_2/A_2^{(m)})^{1-\varepsilon} = p_2^{1-\varepsilon}(A_2^{(m)})^{\varepsilon-1}$ and factor the common denominator term to obtain the stated closed form for t_2'.
3. Compute odds ratios:
$$\frac{t_2}{1-t_2} = \frac{\alpha_2^\varepsilon p_2^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon}}, \qquad \frac{t_2'}{1-t_2'} = \frac{\alpha_2^\varepsilon (p_2/A_2^{(m)})^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon}} = \frac{\alpha_2^\varepsilon p_2^{1-\varepsilon}}{\alpha_1^\varepsilon p_1^{1-\varepsilon}} \cdot (A_2^{(m)})^{\varepsilon-1}.$$
4. Divide the second equality in 3 by the first and take logs to obtain
$$\mathrm{logit}(t_2') - \mathrm{logit}(t_2) = (\varepsilon - 1)\ln A_2^{(m)}.$$
Rearrange to solve for ε. □

Corollary 12.1 (Limiting benchmark cases in the two-good CES gain formula). Under Proposition 11 with $A_2^{(m)} > 0$ and $t_2 \in [0, 1]$, the CES gain formula satisfies the following limits:

1. **Perfect complements (Leontief/Amdahl limit):** as $\varepsilon \to 0$,

$$\frac{V(A')}{V(A)} \to \frac{1}{(1 - t_2) + t_2/A_2^{(m)}}.$$

2. **Cobb–Douglas:** as $\varepsilon \to 1$,

$$\frac{V(A')}{V(A)} \to (A_2^{(m)})^{t_2}.$$

3. **Perfect substitutes:** as $\varepsilon \to \infty$ and $A_2^{(m)} > 1$,

$$\frac{V(A')}{V(A)} \to A_2^{(m)}.$$

Proof

1. Start from Proposition 11:

$$G(\varepsilon) \equiv \left((1 - t_2) + t_2, (A_2^{(m)})^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}}.$$

2. For $\varepsilon \to 0$, note $(A_2^{(m)})^{\varepsilon-1} \to (A_2^{(m)})^{-1}$ and $\frac{1}{\varepsilon-1} \to -1$, hence $G(\varepsilon) \to ((1 - t_2) + t_2/A_2^{(m)})^{-1}$.
3. For $\varepsilon \to 1$, set $r = \varepsilon - 1 \to 0$ and write

$$\ln G(\varepsilon) = \frac{1}{r} \ln\left((1 - t_2) + t_2 e^{r \ln A_2^{(m)}}\right).$$

   Use the expansion $e^{r \ln A} = 1 + r \ln A + o(r)$ to obtain $\ln G(\varepsilon) \to t_2 \ln A_2^{(m)}$, hence $G(\varepsilon) \to (A_2^{(m)})^{t_2}$.
4. For $\varepsilon \to \infty$ with $A_2^{(m)} > 1$, $(A_2^{(m)})^{\varepsilon-1}$ dominates the constant term, so

$$G(\varepsilon) = \left(t_2(A_2^{(m)})^{\varepsilon-1}\left(1 + \frac{1 - t_2}{t_2(A_2^{(m)})^{\varepsilon-1}}\right)\right)^{\frac{1}{\varepsilon-1}} \to (A_2^{(m)}) \cdot t_2^{\frac{1}{\varepsilon-1}} \to A_2^{(m)}.$$

□

Task activation and non-smooth choice (minimal formal extension)

The continuous model above permits corner solutions $t_i(A) = 0$ but remains a convex program. A distinct class of "activation" models introduces discrete feasibility constraints (fixed setup time, unit-demand tasks, lumpy projects). The main implication is potential non-differentiability of $V(\cdot)$ and discontinuous jumps in optimal task selection.

Assumption T1 (Activation costs). Each task (i) has a fixed time cost $f_i \geq 0$ incurred if the task is activated. Let $a_i \in 0, 1$ indicate activation. Feasible allocations satisfy

$$\sum_{i=1}^{n} t_i + \sum_{i=1}^{n} f_i a_i \leq 1, \qquad 0 \leq t_i, \qquad t_i = 0 \text{ if } a_i = 0.$$

Output is $y(A_1 t_1, \dots, A_n t_n)$ as before.

Proposition 13 (Potential non-differentiability under activation). Under Assumption T1, the value function ($V(A)$) (defined analogously to Definition D1 with activation variables) need not be differentiable in (A). In particular, there exist (A, A') such that the set of activated tasks differs between optimizers at (A) and (A'), and at such points the differential formula in Proposition 7 may fail to apply.

Proof

1. Under Assumption T1, the feasible set for ((t, a)) is non-convex because $a \in 0, 1^n$.
2. For non-convex maximization problems, standard envelope theorems that deliver differentiability of the value function may fail at parameter values where the identity of the maximizer changes discontinuously.
3. Choose any instance where two distinct activation patterns $a \neq \tilde{a}$ are both locally optimal for different productivity vectors (e.g., tasks with positive fixed costs and near-ties in the best attainable $y(\cdot)$ across patterns). Then there exists a boundary in (A)-space across which the optimizer switches from (a) to $\tilde{a}$.
4. At such boundaries, ($V(A)$) is the pointwise maximum of finitely many smooth functions (one per activation pattern), hence is generally only directionally differentiable and may fail to be differentiable.
5. Proposition 7 requires differentiability (Assumption A4), which can fail here. □

Summary of derived objects

- **Exact productivity ratio:**
$$\frac{V(A')}{V(A)} = \frac{P(1/A)}{P(1/A')}.$$
- **EV/CV in time units (homogeneous case):**
$$EV = \frac{P(p)}{P(p')} - 1, \quad CV = \frac{P(p')}{P(p)} - 1.$$
- **Differential identity:**
$$d \ln V(A) = \sum_i t_i(A) d \ln A_i.$$
- **Large-change exactness:** integrate compensated (Hicksian) shares along a path.
- **CES closed forms:** unit-expenditure index, shares, two-good gain formula, and elasticity identification from pre/post shares.
- **Bounds from observing baseline or post shares:**
$$\Big(\sum_i t_i/m_i\Big)^{-1} \leq V(A')/V(A) \leq \sum_i t_i' m_i$$
where $m_i = A_i'/A_i$.

Practical summary: estimating $V(A')/V(A)$ from observables

Goal. We want the productivity/output ratio $V(A')/V(A)$ between two states (A baseline and A' post). This is the "gain from changing task productivities".

Key translation. The primal time-allocation problem (Definition D1) is equivalent to a standard expenditure/price-index problem with time prices $p_i = 1/A_i$ (Definitions D2–D4, Propositions 1–2). Under Assumption A3, the gain is exactly a unit-expenditure index ratio (Proposition 3):

$$\frac{V(A')}{V(A)} = \frac{P(p)}{P(p')}, \qquad p = 1/A, \ p' = 1/A'.$$

If you observe only optimal time shares (revealed-preference bounds)

Let $t_i \equiv t_i(A)$ and $t_i' \equiv t_i(A')$ be optimal time shares (Proposition 4), and let $m_i \equiv A_i'/A_i$ be productivity multipliers. Then:

$$\Big(\sum_i \frac{t_i}{m_i}\Big)^{-1} \le \frac{V(A')}{V(A)} \le \sum_i t_i' m_i.$$

- The **lower bound** uses only baseline allocation shares t_i plus measured multipliers m_i. - The **upper bound** uses only post allocation shares t_i' plus measured multipliers m_i.

Intuition: you get a Laspeyres–Paasche style bracket because a baseline allocation is always feasible post-change (and vice versa), so it provides revealed-preference bounds on the cost index (Proposition 4).

If changes are small (local approximation)

Under differentiability (Assumption A4), the log gain has the exact differential form (Proposition 7):

$$d \ln V(A) = \sum_i t_i(A)\, d \ln A_i.$$

For a small discrete change, this implies the first-order approximation (Corollary 7.1):

$$\ln \frac{V(A')}{V(A)} \approx \sum_i t_i \ \ln\Big(\frac{A_i'}{A_i}\Big) = \sum_i t_i \ln m_i.$$

This is a "share-weighted average log multiplier" rule.

If changes are large (exact path integral and practical approximations)

For a large change, the log gain is exactly a path integral of shares times log changes (Proposition 8):

$$\ln \frac{V(A')}{V(A)} = \int_0^1 \sum_i t_i\big(A(\tau)\big)\, \frac{d}{d\tau} \ln A_i(\tau)\, d\tau.$$

If you can approximate the path (or treat baseline and post as endpoints), you get standard Divisia/Törnqvist-style approximations (Proposition 9) using average shares:

$$\ln \frac{V(A')}{V(A)} \approx \sum_i \bar{t}_i \ \ln\Big(\frac{A_i'}{A_i}\Big), \qquad \bar{t}_i \equiv \tfrac{1}{2}(t_i + t_i').$$

If you are willing to assume CES (closed forms and identification)

Under the CES aggregator (Assumption C1), you get: - Closed-form **shares** and the unit-expenditure index (Proposition 10). - A simple **two-good gain formula** when only one task's productivity is multiplied (Proposition 11), expressed in terms of the baseline share and the multiplier. - A **share-response identity** that links observed pre/post shares and the multiplier to the elasticity parameter ε (Proposition 12). This can be used to estimate ε from before/after time shares if the multiplier is known.

Welfare in time units (optional interpretation)

If you want welfare analogues, under linear homogeneity the equivalent/compensating variation in "time units" is just a rescaling of the same index ratio (Proposition 5).

Caveat: activation/discrete choice can break the smooth formulas

If tasks have fixed activation costs (Assumption T1), the value function need not be differentiable and share-based differential/path formulas can fail at points where the set of active tasks changes (Proposition 13). In that setting, revealed-preference bounds (like Proposition 4) are typically more robust than differential approximations.

Summary table (what you can say about $V(A')/V(A)$)

Let $m_i \equiv A_i'/A_i$, $m_{\min} \equiv \min_i m_i$, $m_{\max} \equiv \max_i m_i$, and $t_i \equiv t_i(A)$, $t_i' \equiv t_i(A')$.

What you observe / assume	Statement about $\frac{V'}{V}$	Reference
Full model (y known; can solve D1 at A and A')	Exact V'/V by definition	Definition D1
Unit-expenditure indices $P(p), P(p')$ (equivalently can compute them from y)	Exact $V'/V = \frac{P(p)}{P(p')}$ with $p = 1/A$, $p' = 1/A'$	Proposition 3
A fixed time allocation t (not necessarily optimal), and multipliers m	$m_{\min} \le \frac{y(A'\circ t)}{y(A\circ t)} \le m_{\max}$	Proposition 3.1
Only multipliers m (no shares)	$m_{\min} \le V'/V \le m_{\max}$	Corollary 4.1 (part 3)
Baseline shares t and multipliers m (but not t')	$\left(\sum_i \frac{t_i}{m_i}\right)^{-1} \le \frac{V'}{V} \le m_{\max}$	Corollary 4.1 (part 1)
Post shares t' and multipliers m (but not t)	$m_{\min} \le \frac{V'}{V} \le \sum_i t_i' m_i$	Corollary 4.1 (part 2)
Both t and t' plus multipliers m	$\left(\sum_i \frac{t_i}{m_i}\right)^{-1} \le \frac{V'}{V} \le \sum_i t_i' m_i$	Proposition 4
Small changes, know baseline shares t	Approx $\ln \frac{V'}{V} \approx \sum_i t_i \ln m_i$	Corollary 7.1
Large changes, know a path $A(\tau)$ and shares along it	Exact $\ln \frac{V'}{V} = \int_0^1 \sum_i t_i(A(\tau)) \frac{d}{d\tau} \ln A_i(\tau)\, d\tau$	Proposition 8
CES (Assumption C1), $n = 2$, only task 2 multiplied by $A_2^{(m)}$, know baseline t_2	Exact $\frac{V'}{V} = \left((1 - t_2) + t_2 (A_2^{(m)})^{\varepsilon - 1}\right)^{\frac{1}{\varepsilon - 1}}$	Proposition 11
CES (Assumption C1), $n = 2$, observe $t_2, t_2', A_2^{(m)}$	Identify $\varepsilon = 1 + \frac{\mathrm{logit}(t_2') - \mathrm{logit}(t_2)}{\ln A_2^{(m)}}$ (then use Prop 11 for $\frac{V'}{V}$)	Proposition 12 (+ Proposition 11)